

Tsallis information dimension of complex networks

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Abstract

The fractal and self-similarity properties are revealed in many complex networks. In order to show the influence of different part in the complex networks to the information dimension, we have proposed a new information dimension based on Tsallis entropy namely Tsallis information dimension. The Tsallis information dimension can show the fractal property from different perspective by set different value of q .

Keywords: Complex networks, Information dimension, Tsallis entropy, Tsallis information dimension

1. Introduction

The complex networks have been applied in many disciplines [1, 2, 3, 4, 5, 6, 7]. Researchers have revealed several properties of the complex networks, such as small-world phenomena [8], scale-free degree [9], fractal, self-

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similarity and community structure [10],etc. The fractal and self-similarity properties have shown the structure characteristic of the complex networks, many researchers have been attracted to explore it [11, 12, 13, 14, 15, 16, 17]. In order to describe the fractal properties, Song *et.al* proposed the dimension of the complex networks [18, 19, 20].

Recently, an information dimension of the complex networks has been proposed by Wei *et.al* in [21]. In the information dimension, the boxes which contain more nodes have a maximum effect to the information dimension. However, sometimes those boxes contain few nodes may play an important role in the fractal property. In order to show the influence of the boxes which have different mounts of nodes to the information dimension. A new information dimension based on Tsallis entropy [22] is proposed in this paper. In the proposed method, setting different values of q means chose different part as the main effect of the information dimension.

The rest of this paper is organised as follows. Section 2 introduces some preliminaries of this work. In section 3, a new information dimension of complex networks based on the Tsallis entropy is proposed. The application of the proposed method is illustrated in section 4. Conclusion is given in Section 5. .

2. Preliminaries

2.1. Box-covering algorithm of complex networks

Song *et.al* have proposed a new box-covering algorithm for complex networks [20, 19, 18]. It contains a new definition for the box size l_B which is

based on the distances between the nodes in the complex networks.

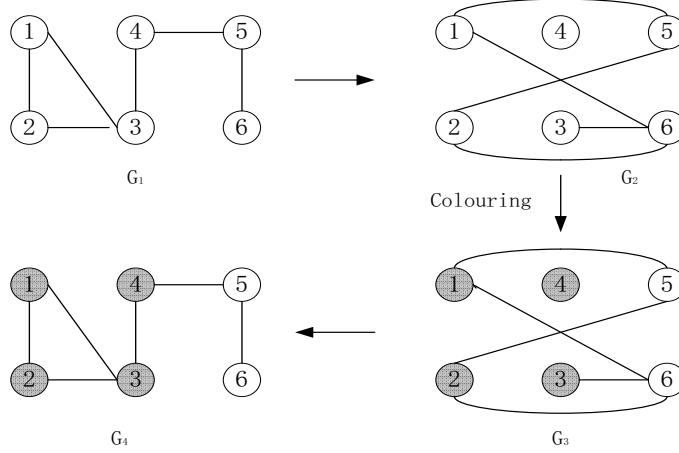


Figure 1: The classical box-covering algorithm for complex networks, where $l = 3$. The network G_1 is original network with 6 nodes and 6 edges. The network G_2 is obtained by only connecting to nodes which distance between them not less than 3 in network G_1 . The network G_3 is obtained when the greedy algorithm is used for node coloring on G_2 [20].

For a given network G_1 and box size l_B , a box is a set of nodes where all distances l_{ij} between any two nodes i and j in the box are smaller than l_B . The minimum number of boxes required to cover the entire networks is denoted by N_B .

For $l_B=1$, N_B is obviously equals to the size of the network N , while $N_B=1$ for $l_B \geq l_B^{\max}$, where l_B^{\max} is the diameter of the network plus one, the diameter of the network equals to the maximum distance in the network [20].

If the distance between two nodes in G_1 is greater than l_B , these two neighbors cannot belong in the same box. According to the construction of

G_2 , these two nodes will be connected in G_2 and thus they will not belong in the same box in G_1 . On the contrary, if the distance between two nodes in G_1 is less than l_B , it is possible that these nodes belong in the same box. In G_2 these two nodes will not be connected and it is allowed for these nodes to carry the same color, it will belong to the same box in G_1 [20]. More details are shown in Fig. 1.

The box-covering algorithm is used to calculate the minimum numbers of box N_l by Song *et.al.* Then the fractal dimension d_B of the complex networks can be described by the relationship between N_l and l_B . The details are shown in Eq. (1).

$$d_B = - \lim_{l_B \rightarrow 0} \frac{\ln N_l}{\ln l_B} \quad (1)$$

2.2. Tsallis entropy

The entropy is defined by Clausius for thermodynamics [23], connects the macroscopic and microscopic worlds. For a finite discrete set of probabilities the definition of the Boltzmann-Gibbs [?] entropy is given as follows:

$$S_{BG} = -k \sum_{i=1}^N p_i \ln p_i \quad (2)$$

Where BG stands for *Boltzmann – Gibbs*, the S_{BG} represents the Boltzmann-Gibbs entropy. The conventional constant k is the Boltzmann universal constant for thermostistical systems, the value of k will being taken to be unity in information theory [24, 25].

In 1988, a more general form for entropy have been proposed by Tsallis [22]. It is shown as follows:

$$S_q = -k \sum_{i=1}^N p_i \ln_q p_i \quad (3)$$

The q – *logarithmic* function in the Eq. (3) is presented as follows [24]:

$$\ln_q p_i = \frac{p_i^{1-q} - 1}{1 - q} (p_i > 0; q \in \Re; \ln_1 p_i = \ln p_i) \quad (4)$$

Based on the Eq. (4), the Eq. (3) can be rewritten as follows:

$$S_q = k \frac{1 - \sum_{i=1}^N p_i^q}{q - 1} \quad (5)$$

Where N is the number of the subsystems.

2.3. Information dimension

Based on the information entropy and the box-covering algorithm, an information dimension has been proposed by Wei *et.al* in [21].

The information of the complex networks is shown as follows:

$$I = - \sum_{i=1}^{N_b} p_i \ln(p_i) \quad (6)$$

The p_i in the Eq. (6) represents the probability of the nodes in the i th box. It is shown in Eq. (7).

$$p_i = \frac{n_i}{n} \quad (7)$$

Where n_i is the node number in the i th box, n is the total number of the nodes in the complex networks [21].

Depends on the relationship between information of the complex networks and the box size. The information dimension of the complex networks is shown in Eq. (8) [21].

$$d_b = -\lim_{l \rightarrow 0} \frac{I}{\ln l} = \lim_{l \rightarrow 0} \frac{\sum_{i=1}^{N_b} p_i \ln(p_i)}{\ln l} \quad (8)$$

Where d_b is the information dimension of the complex network. Based on Eq. (6), the Eq. (8) can be rewritten as follows:

$$d_b = \lim_{l \rightarrow 0} \frac{\sum_{i=1}^{N_b} \frac{n_i(l)}{n} \ln(\frac{n_i(l)}{n})}{\ln l} \quad (9)$$

3. Tsallis information dimension

In this section, a Tsallis information dimension of the complex networks, d_T , is proposed as follows:

$$d_T = \frac{\frac{1 - \sum_{i=1}^N p_i(l)^q}{q-1}}{\ln l} (q \in \Re) \quad (10)$$

Where l is the box size in the box-covering algorithm. The numerator is the Tsallis entropy which is defined in Eq. (5). It can be easily seen that when $q = 1$ the Tsallis information dimension is degenerated to the information dimension of complex networks in [21].

Similar to Shannon's information volume, we use the Tsallis entropy to define the information volume of complex networks as follows,

$$I_v = \frac{1 - \sum_{i=1}^{N_p} p_i^q}{q - 1} \quad (11)$$

We discuss the relationship between the parameter q and the information dimension of the complex networks.

CASE 1, when $q \rightarrow -\infty$, the boxes with the minimum probability have the maximum effect on the information dimension of the complex networks.

CASE 2, when $q \rightarrow 0$, the boxes with different probability have the same effect on the information dimension of the complex networks.

CASE 3, when $q \rightarrow 1$, the Tsallis information dimension is degenerated to the information dimension in [21].

CASE 4, when $q \rightarrow \infty$, the boxes with the maximum probability have the maximum effect on the information dimension of the complex networks. The information dimension of the complex networks is closed to 0.

It can be easily found that, with the increase of q , the information dimension of the complex networks is decreased.

4. Application

In this section, we use the proposed method to calculate the information dimension of four real networks, namely, the US-airlines networks [26], Email networks [26] and the Germany highway networks [27]. The results are given in Table 1.

The comparison between the information dimension in [21] and the proposed information dimension is shown in Table 2.

Table 1: Tsallis information dimension of real networks

	Germany highway [27]	Us-airline [26]	Email [26]
Nodes	1168	500	1133
edges	2486	5962	10902
$d_{T(q=0.1)}$	61.88	62.89	175.20
$d_{T(q=0.5)}$	10.646	15.780	23.778
$d_{T(q=1.0)}$	1.9384	2.9682	3.5132
$d_{T(q=1.5)}$	0.66732	1.0585	1.1131
$d_{T(q=2.0)}$	0.35145	0.5758	0.5817
$d_{T(q=10)}$	0.0268	0.0564	0.0562
$d_{T(q=100)}$	0.0009	0.0037	0.0039
$d_{T(q=1000)}$	0.0003	0.0002	0.00029

Table 2: The results with different methods

Networks	Nodes	edages	d_b	$d_{T(q=0.1)}$	$d_{T(q=1)}$	$d_{T(q=1000)}$
Germany highway [27]	1168	2486	1.9384	61.8816	1.9384	0.00030
Us-airline [26]	500	5962	2.9682	62.8919	2.9682	0.00020
Email [26]	1133	10902	3.5132	175.21	3.5132	0.00029

In the Table 1 and Table 2, the d_b represents the information dimension of the complex networks which is calculated by the method in [21]. The $d_{T(q=x)}$ represents the information dimension of complex networks which is calculated by the proposed method.

The slope of the straight lines in the Figure (2-4) represents the information dimension of complex network. The results have shown that the

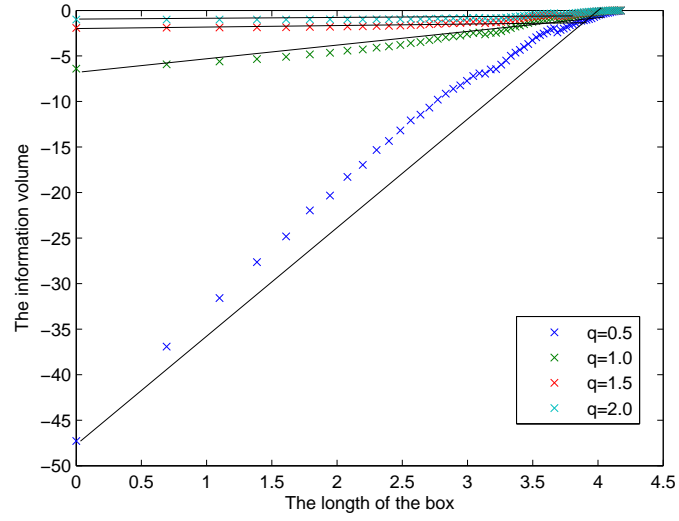


Figure 2: The Germany highway network

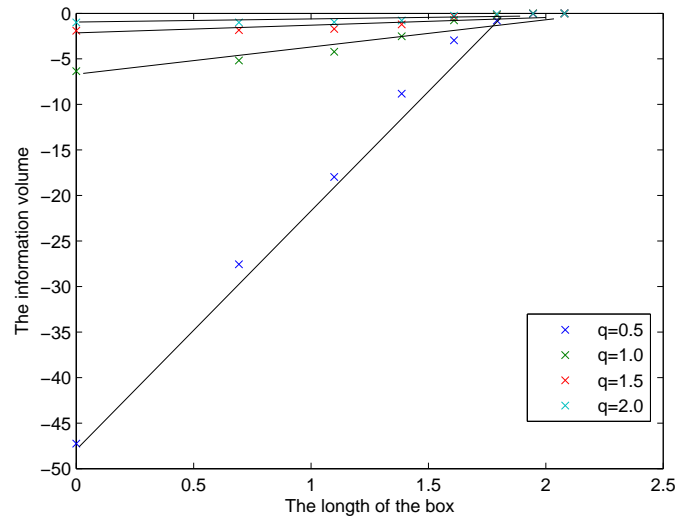


Figure 3: The Email network

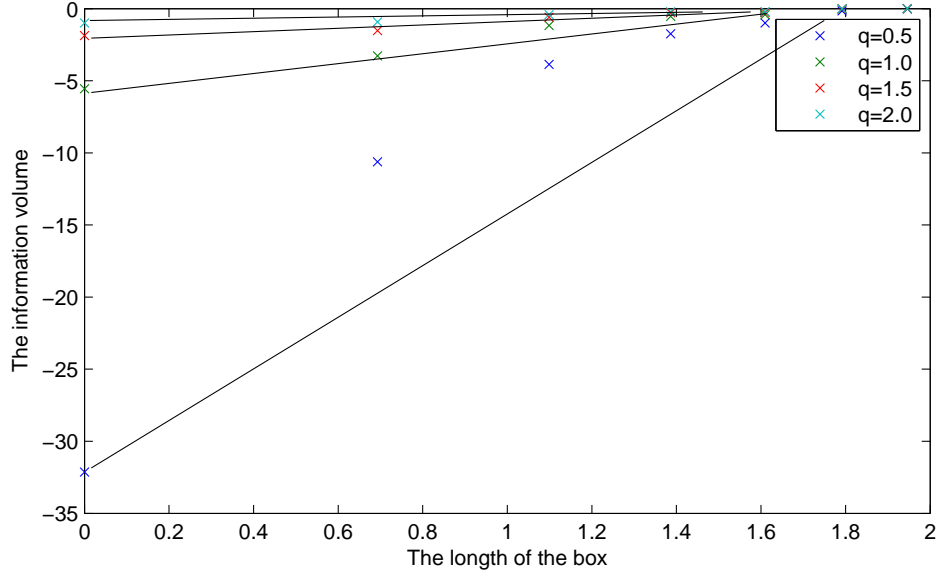


Figure 4: The US-airport network

information dimension is in inverse proportion to the value of q .

5. Conclusion

The information dimension is widely used to illuminate the fractal and self-similarity properties of the complex networks. In this article, a general method to calculate the information dimension of complex networks has been proposed based on the Tsallis entropy. It can be used to describe the influence of different parts in the complex networks to the fractal property. The proposed Tsallis information dimension is a generalization of the existing information dimension to the complex networks.

Acknowledgments

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